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A SET OF SIMPLE, ACCURATE EQUATIONS FOR CIRCULAR CYLINDRICAL ELASTIC SHELLS*

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Abstract—The Sanders' equations for a circular cylindrical elastic shell of constant thickness are reduced to a single, simple, fourth order partial differential equation for the complex-valued function $W+i\sqrt{(A/D)}F$, where W is the midsurface normal deflection, F is a stress function, and A/D is an elastic constant. Auxiliary equations expressing the tangential midsurface displacements, stress resultants, stress couples, and Kirchhoff edge forces in terms of W, F, and surface load integrals are also derived. Approximations are introduced only in the stress–strain relations, but the resulting errors are shown to be negligible by Koiter's arguments. Work of previous writers is reviewed and compared with results of the present paper.

1. INTRODUCTION

CIRCULAR cylindrical elastic shells of constant thickness, because they are technically important and easy to analyze mathematically, and because they exhibit nearly every type of behavior found in shells of more complicated geometry, have been extensively investigated throughout the history of shell theory, especially within the last thirty-five years. The technical uses of circular cylindrical shells are too well known to be catalogued here. Their mathematics is simple because of their simple midsurface geometry which makes their governing equations, in lines of curvature coordinates, of the constant coefficient type. The many important phenomena displayed by the equations of circular cylindrical shells, such as boundary layers, the degeneracy of boundary layers near edges which coincide or nearly coincide with midsurface asymptotic lines, the inadequacy of equating the two in-plane shear stress resultants, or the limitations of the assumption that the "interior" behavior of the shell is the sum of a membrane and an inextensional bending state, make circular cylindrical shells ideal for testing the adequacy of simplifications proposed in general shell theory.

In this paper we propose a new set of equations for the linear behavior of elastically isotropic, constant thickness, circular cylindrical shells subject to edge and surface loads. The final form of our equations consists of a single, non-homogeneous, fourth order partial differential equation for a complex-valued displacement–stress function, Ψ, together with auxiliary equations for midsurface displacements, stress resultants, stress couples, and effective Kirchhoff edge forces. The chief virtue of these new equations, as compared to others which have been proposed,† is that they are at once *concise* and *adequate*. By *adequate*, we mean that for any given set boundary conditions, the solution

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† With the exception of Novozhilov's [1], which we discuss in Section 2.

of the unreduced equations of any of the acceptable first approximation shell theories* will agree with the solutions of our equations to within errors inherent in the stress–strain relations of the first approximation theories themselves, namely, to within errors of $O(h/a)$ where h is the shell thickness and a the midsurface radius.

Our derivation starts from a set of equations for arbitrary shells first proposed by Sanders [3] in 1959. (An improved derivation of these equations, employing an *exact* definition of the modified symmetric shear stress resultant, is given by Budiansky and Sanders in [4].) Utilizing Koiter's arguments [2] on the adequacy of, and the errors in, Love's uncoupled stress–strain relations, we reduce the Sanders' equations for a circular cylindrical shell to two coupled fourth order partial differential equations for the midsurface normal deflection W and a stress function F. One of these equations has a nonhomogeneous part involving the surface loads and their integrals. In the reduction, the static geometric analogy enjoyed by the Sanders' equations is preserved, which enables us to combine the two equations for W and F into a single equation for a complex displacement–stress function Ψ. We further show that all auxiliary variables, or in some cases the partial derivatives of these variables, can be expressed in terms of W, F, and load integrals alone.

The claim that our reduced equations are adequate is based on the fact that we make approximations only in those parts of the governing equations into which it is necessary to introduce *stress–strain relations*—namely, the bending terms in the equilibrium equations and the extensional strain terms in the compatibility equations—and that the approximations involve only neglect of terms of the type M/a compared to N or neglect of terms of the type ε/a compared to $\varkappa$, where M, N, ε, and $\varkappa$ are, respectively, typical stress couples, stress resultants, extensional and bending strains. This means, first, that the errors we introduce into the stress–strain relations are consistent with the errors already contained in these relations because of the neglect of transverse shearing and normal stress effects [2, 5]; and second, that for the extreme states of inextensional bending and pure membrane stress, where it is known that *indiscriminate* neglect of $O(h/a)$ terms in the governing equations can lead to errors of $O(1)$ in the final solutions,† our equations will lead to solutions with errors of only $O(h/a)$.

A summary of our final equations may be found in Section 7.

2. SIGNIFICANT DEVELOPMENTS IN THE HISTORY OF CIRCULAR CYLINDRICAL SHELLS

To place our results in perspective, we have listed in this section various sets of reduced equations which, in our opinion, have marked a significant development in the theory of circular cylindrical shells. No attempt has been made to indicate the method of derivation of these equations, nor have equations for auxiliary quantities (except in a few instances) or boundary conditions been listed, although these are certainly as important as the reduced equations themselves. Also, for simplicity, surface loads terms have been omitted. Shell geometry and sign conventions for displacements, loads, stress-resultants, and stress couples are indicated in Fig. 1. Below, and elsewhere in this paper, primes and dots denote, respectively, differentiation with respect to the nondimensional axial distance $\xi = az$, and the angular variable θ.

* As defined by Koiter [2].

† We cite an example of this in Section 2.

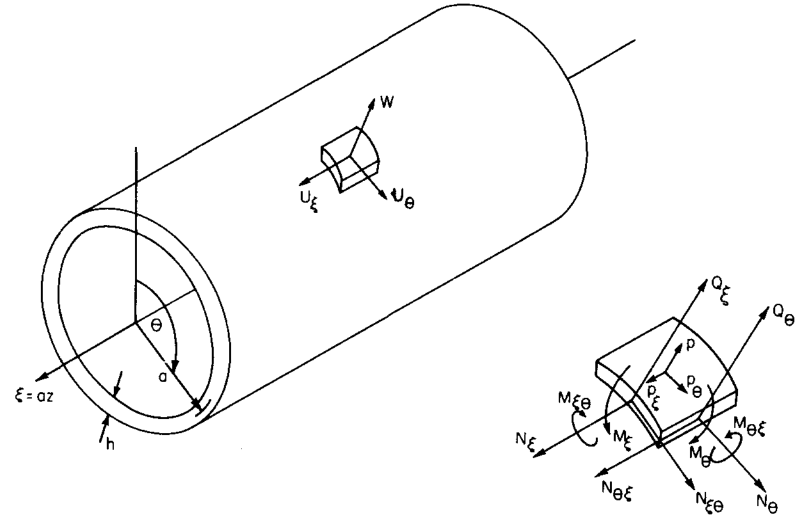

FIG. 1. Geometrical and stress conventions.

The first set of cylindrical shell equations general enough to include all possible states of (linear) deformation, yet simple enough to yield manageable solutions, appear to have been given by Love in the third edition (1920) of his treatise [6, pp. 574 ff.]. (Also, [7, pp. 582 ff.].) From the three exact force equilibrium equations expressed in terms of stress resultants and couples, Love obtained, via a set of stress–strain and strain displacement relations, three simultaneous equations for the midsurface displacements. In our notation, Love's equations read

$$U_\xi'' + \frac{1-\nu}{2} U_\xi^{\cdot\cdot} + \frac{1+\nu}{2}\left[1 - \frac{1}{12}\frac{1-\nu}{1+\nu}\left(\frac{h}{a}\right)^2\right] U_\theta'^{\cdot} + \nu W' + \frac{1-\nu}{24}\left(\frac{h}{a}\right)^2 W'^{\cdot\cdot} = 0 \tag{2.1}$$

$$\frac{1-\nu}{2}\left[1+\frac{1}{4}\left(\frac{h}{a}\right)^2\right]U_\theta'' + \left[1+\frac{1}{12}\left(\frac{h}{a}\right)^2\right]U_\theta^{\cdot\cdot} + \frac{1+\nu}{2}U_\xi'^{\cdot} + W^{\cdot} - \frac{1}{12}\left(\frac{h}{a}\right)^2\left(\frac{3-\nu}{2}W'' + W^{\cdot\cdot}\right)^{\cdot} = 0 \tag{2.2}$$

$$W + U_\theta^{\cdot} + \nu U_\xi' + \frac{1}{12}\left(\frac{h}{a}\right)^2 [\nabla^4 W - (2-\nu)U_\theta''^{\cdot} - U_\theta^{\cdot\cdot\cdot}] = 0 \tag{2.3}$$

where ν is Poisson's ratio.

It seems curious that, despite the renown of Love's treatise, most writers credit Flügge [8] (1932) with having obtained the first adequate, workable, set of circular cylindrical shell equations. Certainly the well-known texts of Flügge [9], Timoshenko and Woinowsky-Krieger [10], Novozhilov [1], Vlasov [11], and Goldenveiser [12], as well as the two fundamental papers of Donnell [13, 14], make no *specific* mention of the above-cited equations of Love. This oversight is probably explained by the fact that one generally ascribes to Love a set of equations based on his first-approximation theory [7, p. 531] which assumes that the two in-plane shear stress resultants, $N_{\xi\theta}$ and $N_{\theta\xi}$, are equal.* However, in his derivation of (2.1) to (2.3), Love *distinguished* between $N_{\xi\theta}$ and $N_{\theta\xi}$,

* An example of the non-negligible errors this assumption can introduce is given by Reissner [15, 16].

obtaining an expression for $N_{\xi\theta}+N_{\theta\xi}$ from the stress–strain relations and an expression for $N_{\xi\theta}-N_{\theta\xi}$ from the moment equilibrium equation about the normal.

It should be emphasized that, in general, terms of relative order $(h/a)^2$ in (2.1) to (2.3) *cannot* be neglected even though terms of relative order (h/a) were neglected in the derivation of the stress–strain relations used in obtaining (2.1) to (2.3). To cite an example, if we set $U_\xi = (\ )' = 0$, (2.1) to (2.3) reduce, as they should, to the two equations of ring bending of plane strain theory. If U_θ is now eliminated between these two equations, the terms independent of (h/a) identically cancel, and the following equation for W is obtained.

$$(W^{\cdot\cdot\cdot\cdot}+2W^{\cdot\cdot}+W)^{\cdot} = 0. \tag{2.4}$$

Had the underlined term in (2.2) been omitted as being of relative order $(h/a)^2$, then the last term in (2.4), which is non-negligible, would have erroneously been found to be zero.

This importance of apparently negligible terms in (2.1) to (2.3), which is by no means unique to the Love equations, is closely related to problems of inextensional and partially inextensional deformation, and is one of the chief drawbacks in taking the midsurface displacements as the dependent variables. A great advantage of the dual displacement–stress function approach used to derive the new set of equations proposed in the present paper is that this small-term problem is completely avoided.

The popular Flügge equations [8], [9, p. 219],* in our notation, read

$$U_\xi''+\frac{1-\nu}{2}\left[1+\frac{1}{12}\left(\frac{h}{a}\right)^2\right]U_\xi^{\cdot\cdot}+\frac{1+\nu}{2}U_\theta'^{\cdot}+\nu W'+\frac{1}{12}\left(\frac{h}{a}\right)^2\left(\frac{1-\nu}{2}W^{\cdot\cdot}-W''\right)^{\cdot} = 0 \tag{2.5}$$

$$\frac{1-\nu}{2}\left[1+\frac{1}{4}\left(\frac{h}{a}\right)^2\right]U_\theta''+U_\theta^{\cdot\cdot}+\frac{1+\nu}{2}U_\xi'^{\cdot}+W^{\cdot}-\frac{3-\nu}{24}\left(\frac{h}{a}\right)^2 W''^{\cdot} = 0 \tag{2.6}$$

$$W+U_\theta^{\cdot}+\nu U_\xi'+\frac{1}{12}\left(\frac{h}{a}\right)^2\left(\nabla^4 W+2W^{\cdot\cdot}+\underline{W}+\frac{1-\nu}{2}U_\xi'^{\cdot\cdot}-U_\xi'''-\frac{3-\nu}{2}U_\theta''^{\cdot}\right) = 0. \tag{2.7}$$

Note that the terms in (2.5) to (2.7) proportional to $(h/a)^2$ are considerably different from the corresponding terms in (2.1) to (2.3). In particular, the term of relative order $(h/a)^2$ which must be kept in order to obtain the equations of ring bending—the underlined term in (2.7)—now appears in a different place and in a different form than it did in Love's equations.

A significant simplification of Flügge's equations was proposed by Donnell [13] in 1933 in conjunction with an analysis of torsional buckling. By omitting a number of terms in Flügge's equations, Donnell was able to obtain the single eighth order equation,

$$\nabla^8 W+4\mu^4 W'''' = 0 \tag{2.8}$$

where

$$4\mu^4 = 12(1-\nu^2)(a/h)^2 \tag{2.9}$$

is a large parameter which appears constantly throughout the rest of this paper. As Donnell himself indicated [13], (2.8) is generally valid only if the deformation pattern has a characteristic circumferential wavelength small compared to the radius a. The fact that (2.8) does not include the ring bending equation (2.4) as a special case is evidence of this limitation.

* The original papers of Flügge [8] and Donnell [13, 14] were concerned primarily with buckling problems, and their equations contain a number of non-linear terms. Any references in this paper are to the linear parts of these equations.

To obtain a more accurate equation than (2.8), Donnell [14] in 1938 started with a set of shell equations in which he attempted, at the start, "to include all terms which might be significant". He then reduced these equations to a single equation for W without neglecting any terms along the way and attempted to ascertain which terms in this single equation could be neglected. The "modified" or "extended" equation obtained in this fashion was

$$\nabla^8 W + 2W^{\cdot\cdot\cdot\cdot\cdot\cdot} + W^{\cdot\cdot\cdot\cdot} + 4\mu^4 W'''' = 0 \tag{2.10}$$

which differs from (2.8) only by the addition of two terms.

Although (2.10) now includes the ring bending equation (2.4) as a special case, Dr. V. T. Buchwald has pointed out to me that the extended Donnell equation contains another limitation in that it leads to an *incorrect* overall moment–displacement relation for a very long cantilevered circular cylindrical shell acted upon by a net moment at its free end. Further comment on this point will be found in a footnote at the end of Section 5.

In 1958 Morley [17], seeking an equation which retained the accuracy of Flügge's equations* but the simplicity of Donnell's equation (2.8), proposed the equation

$$\nabla^4(\nabla^2+1)^2 W + 4\mu^4 W'''' = 0. \tag{2.11}$$

Morley's equation contains several notable improvements over Donnell's extended equation (2.10). First, the necessarily invariant nature of the equation for W is more evident. Second, (2.11) contains both ring *and* beam bending as special cases. And third, (2.11) can be factored into the form

$$[\nabla^2(\nabla^2+1) + i2\mu^2\partial^2/\partial\xi^2][\nabla^2(\nabla^2+1) - i2\mu^2\partial^2/\partial\xi^2]W = 0 \tag{2.12}$$

which, among other things, tremendously simplifies calculation of the roots of the characteristic polynomials which arise from solving (2.11) by separation of variables.

The numerous contributions of Soviet writers to the theory of cylindrical shells is outlined in chapter III of Novozhilov's book [1]. We mention here two important, and relevant equations. According to Novozhilov [1, p. 90], Feinburg in 1936 proposed a simplified equation of the form

$$\nabla^4\Psi - i2\mu^2\Psi'' = 0 \tag{2.13}$$

where Ψ is a complex-valued displacement–stress function defined by

$$\Psi = W + i(2\mu^2/Eah)F. \tag{2.14}$$

The new symbols appearing in (2.14) are F, the Airy stress function of plane stress theory and E, Young's modulus. Equation (2.13) will be recognized as nothing more than the basic equation of shallow shell theory specialized to a cylinder.† Upon elimination of F, (2.13) reduces to the simplified Donnell equation (2.8), and thus suffers from the same limitations as this latter equation. Nevertheless, there are good reasons why it is preferable to work with Feinburg's equation instead of Donnell's. First, (2.13) emphasizes the basic duality among the field equations of shell theory known as the static-geometric analogy (of which more shall be said later). Second, as a consequence of the static-geometric analogy, the order of (2.13) is, effectively, half that of (2.8). This is especially useful in simplifying the algebra in those cases where the boundary conditions can be expressed

* By this time, Flügge's equations had been reduced to a single equation for W.

† Of course, at the time, Marguerre's general theory of shallow shells [18] had not appeared.

in terms of W and F alone (e.g. see [19]). And third, by working with W and F (i.e. Ψ) instead of W alone, a number of auxiliary formulas are greatly simplified. For example, when using the simplified Donnell equations, the only way to express the axial stress resultant N_ξ in terms of W alone is to write

$$a\nabla^4 N_\xi = -(1-\nu^2)EhW''^{\cdot\cdot} \tag{2.15}$$

whereas, using the Feinburg equations, one has, simply,

$$a^2 N_\xi = F^{\cdot\cdot}. \tag{2.16}$$

In 1946 Novozhilov [1, p. 184] proposed an equation for cylindrical shells of arbitrary cross-section which, specialized to circular cross sections, reads in our notation

$$\nabla^4 \tilde{T} + \tilde{T}^{\cdot\cdot} - i2\mu^2 \tilde{T}'' = 0 \tag{2.17}$$

where

$$\tilde{T} = N_\xi + N_\theta - i(Eha/2\mu^2)(\varkappa_\xi + \varkappa_\theta) \tag{2.18}$$

and $\varkappa_\xi$ and $\varkappa_\theta$ are bending strains. Note that, aside from the different dependent variable, (2.17) differs in form from (2.13) only by the addition of a single term, yet because of this term, (2.17) is applicable to both ring and beam bending, though (2.13) is not.

Despite its compactness and comprehensiveness, Novozhilov's equation has not received much attention in the Western literature. One reason is the relatively recent translation date of his book (1959). Another may be that, in deriving (2.17), Novozhilov begins by specializing to cylindrical shells, a set of equilibrium-compatibility equations for arbitrary shells [1, equations (16.10)] into which he has introduced the assumption that the in-plane shear stress resultants are equal [1, equation (16.4)]. However, it turns out that, for circular cylindrical shells at least, this assumption is unnecessary, if, in equations (40.3) of [1], one takes

$$\tilde{S} = N_{\theta\xi} - i(Eha/2\mu^2)\varkappa_{\theta\xi}. \tag{2.19}$$

Finally, we mention two recent papers by Lukasiewicz [20] and Ichino and Takahashi [21], both of which employ as the basic dependent variables W and an Airy-type stress function F. In [20], Lukasiewicz has attempted to reduce the equations for arbitrary shells to two coupled equations for W and F. For circular cylindrical shells, his equations reduce to

$$D(\nabla^2+1)^2 W - aF'' = 0 \tag{2.20}$$

$$A\nabla^4 F + aW'' = 0 \tag{2.21}$$

where

$$D = \frac{Eh^3}{12(1-\nu^2)} \quad \text{and} \quad A = \frac{1}{Eh} \tag{2.22}$$

Upon elimination of F, (2.20) and (2.21) reduce to Morley's equation (2.11). While one might criticize the lack of symmetry between (2.20) and (2.21), the most important shortcoming in Lukasiewicz's results are his auxiliary equations, which can easily be shown not to be universally applicable. Two of the objectionable ones are

$$N_{\xi\theta} = N_{\theta\xi} \qquad [20, \text{equation } (3.1)_3] \tag{2.23}$$

and

$$M_{\theta\xi} = D(1-\nu)W'^{\cdot} \qquad [20, \text{ equations } (3.1)_6 \text{ and } (5.1)_3]. \tag{2.24}$$

Reissner's analysis of the split tube under torsion [15, 16] shows that (2.23) is unacceptable, and it is not difficult to construct another example to show that (2.24) is generally incorrect.

More satisfactory results have been obtained by Ichino and Takahashi [21]. Starting with the exact equilibrium equations and the approximate Flügge stress–displacement relations, these authors arrived at two coupled equations for W and F which they were able to combine into

$$\nabla^2(\nabla^2+1)\Psi+\frac{i}{2}(1-\nu)\mu^{-2}\nabla^2(\Psi^{\cdot\cdot}+\Psi-\Psi'')-i2\mu^2\Psi'' = 0 \tag{2.25}$$

where Ψ is given by (2.14). Furthermore, in their auxiliary equations, they distinguished between $N_{\xi\theta}$ and $N_{\theta\xi}$ and used a more accurate equation for $M_{\theta\xi}$.

The new reduced equation for circular cylindrical shells proposed in this paper is

$$\nabla^4\Psi+\Psi^{\cdot\cdot}+\lambda\Psi''-i2\mu^2\Psi'' = 0 \tag{2.26}$$

where Ψ is given by (2.14) and λ can be any arbitrary 0(1) constant. Thus our equation resembles an amalgam of the results of Feinburg, Novozhilov, Lukasiewicz, and Ichino and Takahashi: our complex displacement–stress function Ψ is the same as Feinburg's; the form of (2.26), with $\lambda = 0$, is identical to Novozhilov's (2.17); and we have attempted, as have Lukasiewicz, and Ichino and Takahashi, to extend the use of the basic variables of shallow shell theory, W and F, to non-shallow circular cylindrical shells.

A brief comparison of our equations (2.26) with Novozhilov's (2.17) and Ichino and Takahashi's (2.25) is of interest. The great advantage of Novozhilov's equation is that it easily generalizes to arbitrary cylindrical shells while ours does not.* On the other hand our dependent variable Ψ, being essentially a twice integrated form of Novozhilov's dependent variable $\tilde{T}$, seems more convenient for the application of boundary conditions. Moreover, the form of our equation provides a ready comparison with the standard form of the shallow cylindrical shell equation, (2.13).

Compared with Ichino and Takahashi's equation (2.25), our equation (2.26) is considerably simpler. If our reduction is correct, this implies that the middle group of terms in (2.25) is negligible, as indeed an application of the arguments used in Section 5 of the present paper shows them to be. Also, our auxiliary equations for the remaining dependent variables (which are listed in Section 7) are, in a number of instances, much simpler than the corresponding ones given in [21]. Last, we should point out that use of the modified, symmetric stress resultants of Budiansky and Sanders [4] permits us, without any loss of accuracy, to work with only six stress resultants and couples instead of the eight needed with the Flügge equations used by Ichino and Takahashi.

3. THE SANDERS' EQUATIONS FOR A CIRCULAR CYLINDRICAL SHELL

Specialized to a circular cylindrical shell, the field equations of the Sanders' theory [3, 4], in the notation of Fig. 1, consist of three *exact* reduced force equilibrium equations

* In fact, some unpublished calculations indicate that only the equations of shallow, (nearly) spherical, and (nearly) cylindrical shells can be reduced, without loss of generality, to two coupled fourth order equations for the normal deflection and a stress function.

$$a(N'_\xi + S^{\cdot}) - \tfrac{1}{2}T^{\cdot} + a^2 p_\xi = 0 \tag{3.1a}$$

$$a(S' + N^{\cdot}_\theta) + \tfrac{3}{2}T' + M^{\cdot}_\theta + a^2 p_\theta = 0 \tag{3.1b}$$

$$M''_\xi + 2T'^{\cdot} + M^{\cdot\cdot}_\theta - aN_\theta + a^2 p = 0, \tag{3.1c}$$

six *exact* strain–displacement relations

$$a^2 \varkappa_\xi = -W'', \qquad a^2 \varkappa_\theta = -W^{\cdot\cdot} + U^{\cdot}_\theta \tag{3.2a, b}$$

$$a^2 \tau = -W'^{\cdot} + \tfrac{3}{4}U'_\theta - \tfrac{1}{4}U^{\cdot}_\xi \tag{3.2c}$$

$$a\varepsilon_\xi = U'_\xi, \qquad a\varepsilon_\theta = U^{\cdot}_\theta + W \tag{3.3a, b}$$

$$a\gamma = \tfrac{1}{2}(U'_\theta + U^{\cdot}_\xi), \tag{3.3c}$$

plus a set of *approximate* stress–strain relations which, for an elastically isotropic shell, can be taken in the form*

$$\varepsilon_\xi = A(N_\xi - \nu N_\theta), \qquad M_\theta = D(\varkappa_\theta + \nu\varkappa_\xi) \tag{3.4a, b}$$

$$\varepsilon_\theta = A(N_\theta - \nu N_\xi), \qquad M_\xi = D(\varkappa_\xi + \nu\varkappa_\theta) \tag{3.4c, d}$$

$$\gamma = A(1+\nu)S, \qquad T = D(1-\nu)\tau, \tag{3.4e, f}$$

where A and D are defined by (2.22).

In (3.1) and (3.4), S and T are, respectively, a modified shear stress resultant and a modified twisting stress couple, defined by Budiansky and Sanders [4] in terms of the conventional unsymmetric stress resultants and couples as follows:

$$S = \frac{1}{2}(N_{\xi\theta} + N_{\theta\xi}) + \frac{1}{4a}(M_{\xi\theta} - M_{\theta\xi}) \tag{3.5}$$

$$T = \frac{1}{2}(M_{\xi\theta} + M_{\theta\xi}). \tag{3.6}$$

For a complete system, the above equations must be supplemented by boundary conditions. These may be read off from the expression for the work of the edge loads, $a\Pi_E$. Assume for simplicity that we are dealing with a panel of nondimensional length $\xi = l$ and angular width $\theta = \alpha$. Then, with the displacements satisfying the Kirchhoff hypothesis, we have

$$\begin{aligned}\Pi_E = &\int_0^l [N_\theta U_\theta + S_\theta U_\xi + R_\theta W + M_\theta \varphi_\theta]_{\theta=0}\, d\xi \\ &+ \int_0^\alpha [N_\xi U_\xi + S_\xi U_\theta + R_\xi W + M_\xi \varphi_\xi]_{\xi=l}\, d\theta \\ &+ \int_l^0 [N_\theta U_\theta + \ldots]_{\theta=\alpha}\, d\xi + \int_\alpha^0 [N_\xi U_\xi + \ldots]_{\xi=0}\, d\theta \\ &+ [2TW]_{\substack{\xi=0\\ \theta=0}} + \ldots + [2TW]_{\substack{\xi=0\\ \theta=\alpha}}\end{aligned} \tag{3.7}$$

where

$$\varphi_\xi = -W'/a, \qquad \varphi_\theta = -(W^{\cdot} - U_\theta)/a \tag{3.8}$$

* When we wish to distinguish between (3.4a, c, e) and (3.4b, d, f), we shall refer to the former as the force–extension relations and the latter as the moment–curvature relations.

are the edge rotations and

$$S_\xi = N_{\xi\theta}+M_{\xi\theta}/a, \qquad S_\theta = N_{\theta\xi} \tag{3.9a, b}$$

$$R_\xi = Q_\xi+M^{\cdot}_{\xi\theta}/a, \qquad R_\theta = Q_\theta+M'_{\theta\xi}/a \tag{3.10a, b}$$

are the effective Kirchhoff edge stress resultants. In terms of the Budiansky–Sanders variables, (3.9) and (3.10) can be expressed *exactly* as

$$S_\xi = S+\tfrac{3}{2}T/a, \qquad S_\theta = S-\tfrac{1}{2}T/a \tag{3.11a, b}$$

$$aR_\xi = M'_\xi+2T^{\cdot}, \qquad aR_\theta = M^{\cdot}_\theta+2T'. \tag{3.12a, b}$$

4. COMPATIBILITY CONDITIONS, THE STATIC-GEOMETRIC ANALOGY, AND STRESS FUNCTIONS

While the equations of the preceding section are a complete set, a more symmetric formulation is possible utilizing the Goldenveizer–Lur'e [12] static-geometric analogy.* The static-geometric analogy permits the governing equations to be stated in a concise elegant form, and in many cases (but not all!), the order of these equations is thereby halved. In our reduction of the Sanders' equations for the circular cylindrical shell, the static-geometric analogy shall be exploited fully.

Since the six extensional and bending strains are expressable in terms of the three midsurface displacement components, they cannot be specified independently, but must satisfy compatibility conditions. From (3.2) and (3.3) these follow as

$$a(-\varkappa'_\theta+\tau^{\cdot})+\tfrac{1}{2}\gamma^{\cdot} = 0 \tag{4.1a}$$

$$a(\tau'-\varkappa^{\cdot}_\xi)-\tfrac{3}{2}\gamma'+\varepsilon^{\cdot}_\xi = 0 \tag{4.1b}$$

$$\varepsilon''_\theta-2\gamma'^{\cdot}+\varepsilon^{\cdot\cdot}_\xi+a\varkappa_\xi = 0. \tag{4.1c}$$

If we set $p_\xi = p_\theta = p = 0$ and make the following correspondence of variables (the static-analogy),

$$N_\xi \leftrightarrow -\varkappa_\theta, \qquad N_\theta \leftrightarrow -\varkappa_\xi, \qquad S \leftrightarrow \tau \tag{4.2a, b, c}$$

$$\varepsilon_\xi \leftrightarrow M_\theta, \qquad \varepsilon_\theta \leftrightarrow M_\xi, \qquad \gamma \leftrightarrow -T \tag{4.3a, b, c}$$

then (3.1) and (4.1) become identical.

When the strains are expressed in terms of the displacements, equations (4.1) are identically satisfied. Let particular solutions of (3.1) be given by the surface load integrals of membrane theory. It then follows from the static-geometric analogy that if we introduce the following correspondence between displacements and stress functions

$$W \leftrightarrow F, \qquad U_\xi \leftrightarrow H_\xi, \qquad U_\theta \leftrightarrow H_\theta, \tag{4.4a, b, c}$$

the reduced force equilibrium equations will be identically satisfied if the stress resultants and couples are expressed as follows.

$$a^2N_\xi = F^{\cdot\cdot}-H^{\cdot}_\theta+a^3\int\left[\int (p^{\cdot\cdot}+p^{\cdot}_\theta)\,\mathrm{d}\xi-p_\xi\right]\mathrm{d}\xi \tag{4.5a}$$

$$a^2N_\theta = F''+a^3p \tag{4.5b}$$

* Not all of the linear shell equations proposed in the literature admit a static-geometric analogy. The general form of those which do is given in [4].

$$a^2 S = F'^{\cdot} + \tfrac{3}{4} H_\theta^{\cdot} - \tfrac{1}{4} H_\xi^{\cdot} - a^3 \int (p^{\cdot} + p_\theta)\, \mathrm{d}\xi \tag{4.5c}$$

$$aM_\xi = H_\theta^{\cdot} + F, \qquad aM_\theta = H_\xi' \tag{4.6a, b}$$

$$aT = -\tfrac{1}{2}(H_\theta' + H_\xi^{\cdot}). \tag{4.6c}$$

Stress function representations for the effective Kirchhoff edge stress resultants S_ξ, S_θ, R_ξ, and R_θ are also of interest. These follow from (3.11), (3.12), (4.5) and (4.6) as

$$a^2 S_\xi = -(F' + H_\xi)^{\cdot} - a^3 \int (p^{\cdot} + p_\theta)\, \mathrm{d}\xi \tag{4.7a}$$

$$a^2 S_\theta = -(F^{\cdot} - H_\theta)' - a^3 \int (p^{\cdot} + p_\theta)\, \mathrm{d}\xi \tag{4.7b}$$

$$a^2 R_\xi = F' - H_\xi^{\cdot\cdot}, \qquad a^2 R_\theta = -H_\theta''. \tag{4.8a, b}$$

A further duality among the field equations is exhibited by the stress–strain relations. Observe that if we introduce the correspondence of elastic constants

$$A \leftrightarrow -D, \qquad \nu \leftrightarrow -\nu \tag{4.9a, b}$$

and use (4.2) and (4.3), then the pairs (3.4a, b), (3.4c, d), and (3.4e, f) become identical.

5. REDUCTION OF THE SANDERS' FIELD EQUATIONS

We now proceed, with the aid of certain arguments of Koiter [2], to reduce the Sanders' field equations to two coupled fourth order partial differential equations for the normal midsurface deflection W, and the stress function F. Because of the static-geometric analogy, we shall be able to combine these two equations into a single equation for a complex displacement–stress function Ψ. The reduction is straightforward, and analogous to the one used in shallow shell theory.

We begin with the reduced normal force equilibrium equation,

$$M_\xi'' + 2T'^{\cdot} + M_\theta^{\cdot\cdot} - aN_\theta + a^2 p = 0. \tag{5.1}$$

As noted before, this equation becomes identically satisfied when the stress resultants and couples are expressed in terms of stress functions and load integrals. If instead, we express the stress couples in terms of displacements via the moment–curvature relations (3.4b, d, f) and strain–displacement relations (3.2), but leave N_θ in terms of F and p, then (5.1) can be written

$$D[\nabla^4 W + f(U_\xi, U_\theta, W)] + aF'' = 0 \tag{5.2}$$

where

$$f(U_\xi, U_\theta, W) = \tfrac{1}{2}(1-\nu)U_\xi'^{\cdot\cdot} - \tfrac{1}{2}(3-\nu)U_\theta''^{\cdot} - U_\theta^{\cdot\cdot\cdot}. \tag{5.3}$$

By use of the strain–displacement relations (3.2a) and (3.3), and the compatibility equation (4.1c), we can write

$$f(U_\xi, U_\theta, W) = W^{\cdot\cdot} - a[(3-\nu)\gamma'^{\cdot} - (2-\nu)\varepsilon_\xi^{\cdot\cdot} + \varepsilon_\theta^{\cdot\cdot}]. \tag{5.4}$$

The following, more general form of f is obtained if (4.1c) is multiplied by an arbitrary constant $-\lambda$ and added to (5.4):

$$f(U_\xi, U_\theta, W) = W^{\cdot\cdot} + \lambda W'' - a\underline{[\lambda\varepsilon_\theta'' + (3-2\lambda-\nu)\gamma'^{\cdot} - (2-\lambda-\nu)\varepsilon_\xi^{\cdot\cdot} + \varepsilon_\theta^{\cdot\cdot}]}. \tag{5.5}$$

We now come to the crucial argument in our reduction. We observe that had we started with the set of stress–strain relations

$$M_\xi = D[\varkappa_\xi + \nu\varkappa_\theta - \underline{\lambda\varepsilon_\theta/a}] \qquad (5.6a)$$

$$M_\theta = D[\varkappa_\theta + \nu\varkappa_\xi + \underline{(2-\lambda-\nu)\varepsilon_\xi/a - \varepsilon_\theta/a}] \qquad (5.6b)$$

$$T = D[(1-\nu)\tau - \underline{\tfrac{1}{2}(3-2\lambda-\nu)\gamma/a}], \qquad (5.6c)$$

instead of (3.4b, d, g), then the underlined terms in (5.5) would have been identically zero. Now the stress–strain relations in any first approximation shell theory including Sanders' are obtained from the stress–strain relations (or the strain energy function) of three-dimensional elasticity by invoking the Kirchhoff hypothesis or some equivalent, such as the assumption of a state of three-dimensional plane stress. But Koiter [2] has shown that the errors one introduces into the stress–strain relations of shell theory by the adoption of the Kirchhoff hypothesis are of the same order of magnitude as those one introduces by replacing a bending strain term of the type $\varkappa$ by a term of the type $\varkappa + O(\varepsilon/a)$. Thus, assuming λ to be an arbitrary constant of $O(1)$, we conclude that it is consistent to neglect the underlined terms in (5.5) and (5.6),.and therefore to take (5.2) in the simplified form

$$D(\nabla^4 W + W^{\cdot\cdot} + \lambda W'') + aF'' = 0. \qquad (5.7)$$

To obtain a second equation relating W and F, we give an analogous treatment to the third compatibility equation,

$$\varepsilon_\theta'' - 2\gamma'^{\cdot} + \varepsilon_\xi^{\cdot\cdot} + a\varkappa_\xi = 0. \qquad (5.8)$$

Expressing the extensional strains in terms of stress functions and load integrals via (3.4a, c, e) and (4.5), and setting $a^2\varkappa_\xi = -W''$, we find that (5.8) reduces to

$$A[\nabla^4 F + f(H_\xi, H_\theta, F)] - aW'' = -a^3 AP(p_\xi, p_\theta, p) \qquad (5.9)$$

where

$$P(p_\xi, p_\theta, p) = \nabla^4\left(\iint p \, \mathrm{d}\xi \, \mathrm{d}\xi\right) + \nu p_\xi' - \int p_\xi^{\cdot\cdot} \, \mathrm{d}\xi + (2+\nu)p_\theta^{\cdot} + \iint p_\theta^{\cdot\cdot\cdot} \, \mathrm{d}\xi \, \mathrm{d}\xi \qquad (5.10)$$

and where f is precisely the same function (but with different arguments and ν replaced by $-\nu$) as defined by (5.5).

By virtue of the static-geometric analogy, it follows that Koiter's arguments also imply that the errors we introduce into the force–extension relations (3.4a, c, e) by replacing terms of the type N by terms of the type $N + O(M/a)$ are of the same order of magnitude as the errors already contained in these equations as a consequence of the Kirchhoff hypothesis. Thus we conclude that it is consistent to set

$$f(H_\xi, H_\theta, F) = F^{\cdot\cdot} + \lambda F'', \qquad (5.11)^*$$

whereupon (5.9) reduces to

$$A(\nabla^4 F + F^{\cdot\cdot} + \lambda F'') - aW'' = -a^3 AP(p_\xi, p_\theta, p). \qquad (5.12)$$

* We could choose the arbitrary constant in (5.11) different from the constant λ in (5.5). For symmetry, however, we do not.

Equations (5.7) and (5.12) are the two coupled fourth order equations we set out to derive. They may be expressed in a more concise form by dividing (5.7) by D and then adding to it (5.12) multiplied by $i(AD)^{-\frac{1}{2}}$. This yields the single equation

$$\nabla^4\Psi+\Psi^{\cdot\cdot}+\lambda\Psi''-i2\mu^2\Psi'' = -i2\mu a^2 AP(p_\xi, p_\theta, p) \tag{5.13}$$

where

$$\Psi = W+i\sqrt{(A/D)}F \tag{5.14}$$

and

$$2\mu^2 = a/\sqrt{(AD)} = \sqrt{[12(1-\nu^2)]}a/h. \tag{5.15}$$

A number of remarks are now in order. First, we reiterate that the only place we have introduced approximations is in the stress–strain relations, and that these approximations have been consistent with the approximations inherent in the stress–strain relations of any first approximation shell theory.

Second, even though it is consistent to set $N \approx N+O(M/a)$ and $\varkappa \approx \varkappa+O(\varepsilon/a)$ in the stress–strain relations, this does not necessarily imply that $N \gg O(M/a)$ or $\varkappa \gg O(\varepsilon/a)$. For example, if a state of inextensional bending occurs (such as ring bending), we have, generally, $N = O(M/a)$; consequently, the uncoupled force–extension relations (3.4a, c, e) cease to have any meaning. But this makes sense, for it shows that it is not inconsistent to have zero extensional strains but non-zero stress resultants. Incidentally, the fact that the coefficient of A in (5.9) contains relative errors of $O(1)$ for inextensional bending is inconsequential, since, necessarily, inextensional bending occurs only if the W-term on the left-hand side of (5.9) dominates.

Third, the way in which we have introduced the load integrals is not unique. An alternate way is to define a new stress function

$$F_* = F+a^3\iint \left(p+\int p_\theta \,d\theta\right) d\xi\, d\xi. \tag{5.16}$$

Then (5.7) and (5.12) read

$$D(\nabla^4 W+\underline{W^{\cdot\cdot}+\lambda W''})+aF_*'' = a^4\left(p+\int p_\theta \,d\theta\right) \tag{5.17}$$

$$A(\nabla^4 F_*+\underline{F_*^{\cdot\cdot}+\lambda F_*''})-aW''$$

$$= a^3A\left[\int p_\xi^{\cdot\cdot}\, d\xi+\int p_\theta''\, d\theta-\nu(p_\xi'+p_\theta^{\cdot})+\underline{\iint (p^{\cdot\cdot}+p_\theta^{\cdot})\, d\xi\, d\xi+\lambda\left(p+\int p_\theta\, d\theta\right)}\right]. \tag{5.18}$$

In this form, the reduced equations resemble the equations of shallow shell theory, with the exception of the terms with a dashed underline.

Fourth, our freedom in choosing the constant λ is useful both in simplifying algebra and in comparing our equations with those of other writers. For example, if for a cylindrical shell complete in the θ-direction we assume a product solution of the form

$$\Psi(\xi, \theta) = e^{p\xi}\cos n\theta \qquad n = 0, 1, 2, \ldots \tag{5.19}$$

then the choice $\lambda = 0$ gives the simplest polynomial for p except for $n = 1$, in which case the choice $\lambda = 2$ leads to the simplest polynomial.

To compare (5.13) to other reduced equations which have been proposed, we first set $\lambda = 0$. The homogeneous part of (5.13) in this case is identical in *form* to an equation proposed by Novozhilov. However, as noted in Section 2, the dependent variable in Novozhilov's equation is

$$\tilde{T} = N_\xi + N_\theta - i(Eha/2\mu^2)(\varkappa_\xi + \varkappa_\theta).$$

We now set $\lambda = 1$, write (5.13) as two real equations, and eliminate F between them, obtaining thereby

$$\begin{aligned} &\nabla^4(\nabla^2+1)^2 W + 4\mu^4 W'''' \\ &\quad = (a^4/D)[\nabla^4 p - p_\xi^{\prime\cdot\cdot\cdot} + 2p_\theta^{\prime\prime\cdot} + p_\theta^{\cdot\cdot\cdot} + \nu(p'_\xi + \dot{p}_\theta)''] \end{aligned} \tag{5.20}$$

which is the equation proposed by Morley [17]* on an admittedly *ad hoc* basis.

Finally, let us see if it is possible to reduce our equations to the extended Donnell equation, (2.10). Since preserving the static-geometric analogy is of no concern here, we can obtain more flexibility by taking the arbitrary $O(1)$ constants in (5.7) and (5.13) to be different. Calling the constant in (5.12) λ_*, eliminating F between (5.7) and (5.12), and, for simplicity, setting $P = 0$, we obtain the equation

$$\begin{aligned} &\nabla^8 W + 2W^{\cdot\cdot\cdot\cdot\cdot\cdot} + W^{\cdot\cdot\cdot\cdot} + 4\mu^4 W'''' \\ &\quad + (\lambda+\lambda_*)W'''''' + 2(1+\lambda+\lambda_*)W^{\prime\prime\prime\prime\cdot\cdot} + (4+\lambda+\lambda_*)W^{\prime\prime\cdot\cdot\cdot\cdot} \\ &\quad + (\lambda+\lambda_*)W^{\prime\prime\cdot\cdot} + \lambda\lambda_* W'''' = 0 \end{aligned} \tag{5.21}$$

from which it is clear that *no choice of λ and λ_* will yield the extended Donnell equation (2.10)*.†

* Morley assumed $p_\xi = p_\theta = 0$.

† Equation (5.21) may be used to illustrate why the extended Donnell equation (2.10) leads to an incorrect overall moment–displacement relationship for a very long cantilevered circular cylindrical shell acted upon at its free end by a net moment M. Briefly, by taking the normal deflection in the form

$$W(\xi, \theta) = w(\xi)\cos\theta$$

which is appropriate for such a loading, one may show that, to within terms that are *uniformly negligible for all* ξ, (5.21), plus an appropriate set of boundary conditions, leads to the solution

$$w = \frac{M\xi^2}{2\pi a E h} \tag{A}$$

whereas (2.10), with the same boundary conditions, leads to

$$w = \frac{M\mu^4}{\pi a E h}[\cosh(\xi/\mu^2) - 1]. \tag{B}$$

In particular (A) gives for the tip deflection Δ of a shell of length L

$$\Delta = \frac{ML^2}{2EI}, \qquad I = \pi a^3 h$$

which agrees with the well-known result of elementary beam theory [22, p. 182]. On the other hand (B) gives

$$\Delta = \frac{ML^2}{2EI}\left[1 + \frac{1}{12}\left(\frac{L}{a\mu^2}\right)^2 + \cdots\right]$$

which diverges from the elementary beam theory result without limit as L increases without limit.

6. SIMPLIFIED EQUATIONS FOR DERIVATIVES OF TANGENTIAL DISPLACEMENTS AND STRESS FUNCTIONS

In this section we develop simplified expressions for U'_ξ, $U^{\cdot\cdot}_\xi$, $U^{\cdot}_\theta$, U''_θ and their analogues, H'_ξ, etc., in terms of W, F, and load integrals. It turns out that these expressions for the derivatives of U_ξ, U_θ, H_ξ and H_θ are all that are needed to express any displacement and/or stress boundary conditions in terms of W and F. Thus (5.13) may be solved for Ψ *before* any auxiliary partial differential equations for U_ξ, U_θ, H_ξ, or H_θ are solved.

To obtain the desired expression for U'_ξ, we begin with (3.3a) and (3.4a), combined to read

$$U'_\xi = aA(N_\xi - \nu N_\theta). \tag{6.1}$$

Since (6.1) is a stress–strain relation, we may, according to Koiter's arguments, add terms of the type M/a to the N's.* In particular, we may replace (6.1) by

$$U'_\xi = aA(N_\xi + \underline{M_\xi/a} - \nu N_\theta). \tag{6.2}$$

From (4.5a, b) and (4.6a), it then follows that

$$U'_\xi = (A/a)(F^{\cdot\cdot} + F - \nu F'') + a^2 A\left\{\int\left[\int (p^{\cdot\cdot} + p^{\cdot}_\theta)\, \mathrm{d}\xi - p_\xi\right]\mathrm{d}\xi - \nu p\right\}. \tag{6.3}$$

By the same arguments, one gets from (3.3b), (3.4c), (4.5a, b) and (4.6a),

$$\begin{aligned} U^{\cdot}_\theta &= -W + a\varepsilon_\theta \\ &= -W + aA(N_\theta - \nu N_\xi) \\ &= -W + aA[N_\theta - \nu(N_\xi + \underline{M_\xi/a})] \\ &= -W + (A/a)[F'' - \nu(F^{\cdot\cdot} + F)] + a^2 A\left\{p - \nu\int\left[\int (p^{\cdot\cdot} + p^{\cdot}_\theta)\,\mathrm{d}\xi - p_\xi\right]\mathrm{d}\xi\right\}. \end{aligned} \tag{6.4}$$

For $U^{\cdot\cdot}_\xi$ we get, successively,

$$\begin{aligned} U^{\cdot\cdot}_\xi &= 2a\gamma^{\cdot} - U'^{\cdot}_\theta, \quad \text{by (3.3c)} \\ &= W' + 2a\gamma^{\cdot} - a\varepsilon'_\theta, \quad \text{by (3.3b)} \\ &= W' - aA[N'_\theta - \nu N'_\xi - 2(1+\nu)S^{\cdot}], \quad \text{by (3.4c, e)} \\ &= W' - aA[N'_\theta - \nu N'_\xi - 2(1+\nu)(S - \underline{\tfrac{1}{2}T/a})^{\cdot}] \\ &= W' - aA[N_\theta + (2+\nu)N_\xi]' - 2(1+\nu)a^2 A p_\xi, \quad \text{by (3.1a)} \\ &= W' - aA[N_\theta + (2+\nu)(N_\xi + \underline{M_\xi/a})]' - 2(1+\nu)a^2 A p_\xi \\ &= W' - (A/a)[F'' + (2+\nu)(F^{\cdot\cdot} + F)]' \\ &\quad - a^2 A\left[p' + \nu p_\xi + (2+\nu)\int (p^{\cdot\cdot} + p^{\cdot}_\theta)\,\mathrm{d}\xi\right], \end{aligned} \tag{6.5}$$

by (4.5b) and (6.3). A similar set of substitutions and approximations yields as the final expression for U''_θ,

* Addition of terms of this type to a stress-strain relation will be indicated by an underline.

$$U_\theta'' = -(A/a)[(2+\nu)F''+F^{\cdot\cdot}+F]^{\cdot} \\ -a^2A\Big\{(2+\nu)p^{\cdot}+2(1+\nu)p_\theta+\int\Big[\int(p^{\cdot\cdot}+p_\theta^{\cdot})\,\mathrm{d}\xi-p_\xi\Big]^{\cdot}\,\mathrm{d}\xi\Big\}. \quad (6.6)$$

It is clear that (6.3) and (6.6) may be used to convert displacement and rotation boundary conditions along an edge θ = constant into boundary conditions involving linear combinations of W, F, and their derivatives. The same is true of (6.4) and (6.5) regarding the edge ξ = constant.

By the static-geometric analogy, it follows immediately that

$$H_\xi' = -(D/a)(W^{\cdot\cdot}+W+\nu W'') \quad (6.7)$$

$$H_\theta^{\cdot} = -F-(D/a)[W''+\nu(W^{\cdot\cdot}+W)] \quad (6.8)$$

$$H_\xi^{\cdot\cdot} = F'+(D/a)[W''+(2-\nu)(W^{\cdot\cdot}+W)]' \quad (6.9)$$

$$H_\theta'' = (D/a)[(2-\nu)W''+W^{\cdot\cdot}+W]^{\cdot}. \quad (6.10)$$

An alternate set of equations which in some instances may prove more convenient for solving for U_ξ, U_θ, H_ξ and H_θ, and which follow immediately from (6.3) to (6.10), are

$$\nabla^2U_\xi = [W-(1+\nu)(A/a)(\nabla^2F+F)]'-(1+\nu)a^2A\Big[p'+p_\xi+\int(p^{\cdot\cdot}+p_\theta^{\cdot})\,\mathrm{d}\xi\Big] \quad (6.11)$$

$$\nabla^2U_\theta = -[W+(1+\nu)(A/a)(\nabla^2F+F)]^{\cdot} \\ -(1+\nu)a^2A\Big\{p^{\cdot}+2p_\theta+\int\Big[\int(p^{\cdot\cdot}+p_\theta^{\cdot})\,\mathrm{d}\xi-p_\xi\Big]^{\cdot}\,\mathrm{d}\xi\Big\} \quad (6.12)$$

$$\nabla^2H_\xi = [F+(1-\nu)(D/a)(\nabla^2W+W)]' \quad (6.13)$$

$$\nabla^2H_\theta = -[F-(1-\nu)(D/a)(\nabla^2W+W)]^{\cdot}. \quad (6.14)$$

When (6.3) to (6.10) are substituted into (4.5) to (4.8), equations for stress resultants and couples in terms of W, F, and load integrals are obtained without the need of any further approximations. These equations are listed in the following section. In order to express S_ξ, S_θ, S, and T in terms of W, F, and load integrals alone, it has been necessary to take their first partial derivatives.

7. SUMMARY OF EQUATIONS

Below, we summarize the simplified equations derived in Sections 5 and 6. An "approximately equals" sign, $\approx$, has been used in those equations which, because they involve stress–strain relations, are not exact.

Basic equation

$$\nabla^4\Psi+\Psi^{\cdot\cdot}+\lambda\Psi''-i2\mu^2\Psi'' \\ \approx -i2\mu^2a_*^2A\Big[\nabla^4\Big(\int\int p\,\mathrm{d}\xi\,\mathrm{d}\xi\Big)+\nu p_\xi'-\int p_\xi^{\cdot\cdot}\,\mathrm{d}\xi+(2+\nu)p_\theta^{\cdot}+\int\int p_\theta^{\cdot\cdot\cdot}\,\mathrm{d}\xi\,\mathrm{d}\xi\Big] \quad (7.1)$$

$$\Psi = W+i\sqrt{(A/D)}F, \qquad 2\mu^2 = a/\sqrt{(AD)} = \sqrt{[12(1-\nu^2)]}a/h \quad (7.2\text{a, b})$$

λ = arbitrary, $O(1)$ constant.

Auxiliary equations

Stress resultants

$$a^2N_\xi = F^{\cdot\cdot}+F-aM_\xi+a^3\int\left[\int(p^{\cdot\cdot}+p^{\cdot}_\theta)\,d\xi-p_\xi\right]d\xi$$

$$\approx F^{\cdot\cdot}+F+(D/a)[W''+\nu(W^{\cdot\cdot}+W)]+a^3\int\left[\int(p^{\cdot\cdot}+p^{\cdot}_\theta)\,d\xi-p_\xi\right]d\xi \tag{7.3}$$

$$a^2N_\theta = F''+a^3p \tag{7.4}$$

$$a^2S' \approx -\{F''-(D/a)[W^{\cdot\cdot}+W+\tfrac{1}{2}(3-\nu)W'']\}^{\cdot}-a^3(p^{\cdot}+p_\theta) \tag{7.5a}$$

$$a^2S^{\cdot} \approx -\{F^{\cdot\cdot}+F+(D/a)[W''+\tfrac{1}{2}(1+\nu)(W^{\cdot\cdot}+W)]\}'-a^3\int(p^{\cdot\cdot}+p^{\cdot}_\theta)\,d\xi. \tag{7.5b}$$

Stress couples

$$a^2M_\xi \approx -D[W''+\nu(W^{\cdot\cdot}+W)] \tag{7.6}$$

$$a^2M_\theta \approx -D(W^{\cdot\cdot}+W+\nu W'') \tag{7.7}$$

$$a^2T' \approx -D(1-\nu)W''^{\cdot}, \qquad a^2T^{\cdot} \approx -D(1-\nu)(W^{\cdot\cdot}+W)'. \tag{7.8a, b}$$

Effective Kirchhoff edge stress resultants

$$a^2S^{\cdot}_\xi = -(F^{\cdot\cdot}+F)'+a^2R_\xi-a^3\int(p^{\cdot\cdot}+p^{\cdot}_\theta)\,d\xi$$

$$\approx -\{F^{\cdot\cdot}+F+(D/a)[W''+(2-\nu)(W^{\cdot\cdot}+W)]\}'-a^3\int(p^{\cdot\cdot}+p^{\cdot}_\theta)\,d\xi \tag{7.9}$$

$$a^2S'_\theta = -F'''^{\cdot}-a^2R_\theta-a^3(p^{\cdot}+p_\theta)$$

$$\approx -\{F''-(D/a)[W^{\cdot\cdot}+W+(2-\nu)W'']\}^{\cdot}-a^3(p^{\cdot}+p_\theta) \tag{7.10}$$

$$a^3R_\xi \approx -D[W''+(2-\nu)(W^{\cdot\cdot}+W)]' \tag{7.11}$$

$$a^3R_\theta \approx -D[W^{\cdot\cdot}+W+(2-\nu)W'']^{\cdot}. \tag{7.12}$$

Tangential displacements

$$U'_\xi \approx (A/a)(F^{\cdot\cdot}+F-\nu F'')+a^2A\left\{\int\left[\int(p^{\cdot\cdot}+p^{\cdot}_\theta)\,d\xi-p_\xi\right]d\xi-\nu p\right\} \tag{7.13a}$$

$$U^{\cdot\cdot}_\xi \approx W'-(A/a)[F''+(2+\nu)(F^{\cdot\cdot}+F)]'-a^2A[p'+\nu p_\xi+(2+\nu)\int(p^{\cdot\cdot}+p^{\cdot}_\theta)\,d\xi] \tag{7.13b}$$

$$U^{\cdot}_\theta \approx -W+(A/a)[F''-\nu(F^{\cdot\cdot}+F)]+a^2A\{p-\nu\int\left[\int(p^{\cdot\cdot}+p_\theta)\,d\xi-p_\xi\right]d\xi\} \tag{7.14a}$$

$$U''_\theta \approx -(A/a)[2+\nu)F''+F^{\cdot\cdot}+F]^{\cdot}$$

$$-a^2A\left\{(2+\nu)p^{\cdot}+2(1+\nu)p_\theta+\int\left[\int(p^{\cdot\cdot}+p^{\cdot}_\theta)\,d\xi-p_\xi\right]^{\cdot}d\xi\right\}. \tag{7.14b}$$

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Résumé—Les équations de Sanders d'une enveloppe élastique circulaire cylindrique d'une épaisseur constante sont réduites en une simple, unique équation differentielle partielle de quatrième ordre pour la fonction de valeur complexe $W+i\sqrt{(A/D)}F$, alors que W est la déviation normale de surface moyenne, F est une fonction de tension, et A/D est une constante élastique. Des équations auxiliaires exprimant les déplacements de la surface moyenne tangentielle, les résultantes de tension, les couples de tension et les forces tranchante de Kirchhoff aux termes W, F, et les charges intégrales de surface sont également dérivées. Des approximations sont également introduites uniquement dans les relations contrainte–tension, mais les erreurs en résultant sont constatées négligeables par les arguments de Koiter. Le travail d'autres écrivains est passé en revue et comparé avec les résultats de la présente étude.

Zusammenfassung—Die Sanders' Gleichungen für eine kriesförmige zylindrische elastische Schale von beständiger Dicke sind reduziert zu einer einzelnen einfachen partieller Differentialgleichung vierter Ordnung für die komplexen Funktionen $W+i\sqrt{(A/D)}F$, wo W ist die normale Durchbiegung, F ist eine Beanspruchungs Funktion und A/D ist ein elastischer Festwert. Hilfsgleichungen, welche die tangentialen Verschiebungen, Resultanten, Beanspruchungs momente und Kirchhoffsche Randkräfte durche W, F und Oberflächenbelastungs Integrale ausdrücken werden ebenfalls abgeleitet. Annäherungen sind nur in den Spannungs-Beanspruchungs Beziehungen eingeführt, aber die sich ergebenden Abweichungen zeigen sich als geringfügig nach Koiter's Beweisen. Arbeiten von früheren Verfassern werden besprochen und mit den Ergebnissen der gegenwärtigen Abhandlung verglichen.

Абстракт—Уравнения Сандерса для круглой цилиндрической упругой оболочки с постоянной толщиной сокращены до одиночного простого дифференциального уравнения в частных производных четвёртого порядка для сложно-значной функции $W+i\sqrt{(A/D)}F$, где W представляет из себя срединно-поверхностное нормальное отклонение, F представляет функцию напряжения и A/D—постоянную упругости. Вспомогательные уравнения, выражающие тангенциальные средне-поверхностные смещения, равнодействующие напряжения, пары сил напряжения и силы края Киршхофа в выражениях W, F, и интегралы поверхности нагрубки также выводятся теоретическим путём. Приближения вводятся только в отношениях напряжения-деформации, но результирующие погрешности доказательствами Койтера показаны, как незначительные. Рассматривается работа предыдущих авторов и сравнивается с результатами настоящей статьи.